Atomic Effect Algebras with the Riesz Decomposition Property

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Abstract

We discuss the relationships between effect algebras with the Riesz Decomposition Property and partially ordered groups with interpolation. We show that any σ-orthocomplete atomic effect algebra with the Riesz Decomposition Property is an MV-effect algebras, and we apply this result for pseudo-effect algebras and for states.

Keywords: Effect algebra; Riesz Decomposition Property, MV-effect algebra; interpolation; pseudo-effect algebra; state

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1 Introduction and basic definitions

Effect algebras were introduced by Foulis and Bennett [9] for the study of logical foundations of quantum mechanics. Independently, Kôpka and Chovanec [13] introduced essentially equivalent structures called D-posets. If in an effect algebra $(E;+,0,1)$, we define a partial binary difference operation $-$ as follows: for $a,b \in E$, $a-b=c$ if and only if $b+c$ exists in E and $a=b+c$, then the algebraic system $(E;-,0,1)$ is a D-poset [6]. Effect algebras are a common generalization of several well-established algebraic structures, in particular of orthomodular lattices, orthomodular posets, orthoalgebras and MV-algebras [6].

The most important example of effect algebras is the system $\mathcal{E}(H)$ of all Hermitian operators of a (real, complex or quaternionic) Hilbert space H that are among the zero and the identity operator. $\mathcal{E}(H)$ is used for modeling unsharp observables via POV-measures in measurements in quantum mechanics.

In 1958, Chang [3] introduced MV-algebras to prove the completion of the Łukasiewicz propositional logic [2]. MV-algebras play an important role in many fields of mathematics [2, 6]. Especially, MV-algebras have appeared in effect algebras in many ways: Mundici showed that starting from any AF C*-algebra we can obtain a countable MV-algebra, and conversely, any countable MV-algebra can be derived in such a way [2, 6]. Ravindran [15] proved that Φ-symmetric effect algebras are exactly MV-algebras, and also Boolean D-posets of Chovanec and Kôpka are MV-algebras [6]. Especially, Riečanová [16] has proved that every lattice-ordered effect algebra $(E;+,0,1)$ is an MV-algebra

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$(E;\oplus,',0,1)$ if every pair of elements of the effect algebra E is compatible. Indeed, if we define a binary addition operation $\oplus$ on E as follows, for any $a,b \in E$, $a \oplus b := a + (a' \wedge b)$, and an unary operation $'$ as follows: for any $a \in E$, $a' := 1 - a$. Conversely, in any MV-algebra $(E;\oplus,\,',\,0,1)$, if we define a partial binary addition operation $+$ on E as follows: $a+b$ exists if and only if $a \leqslant b'$ and in such a case $a+b = a \oplus b$, then the algebraic system $(E;+,0,1)$ is a lattice-ordered effect algebra. We recall that any MV-algebra is also called an MV-effect algebra, [11].

Effect algebras with the Riesz decomposition property (RDP) form an important class of effect algebras. An effect algebra with RDP is always an interval in an Abelian partially ordered group [6]. Every MV-effect algebra satisfies RDP. On the other hand, effect algebras with RDP are not necessarily MV-effect algebras. However, every finite effect algebra with the Riesz decomposition property is an MV-effect algebra [1]. In this paper, we will continue in the study of the conditions when effect algebras with RDP are MV-effect algebras.

The paper is organized as follows. In Section 2, we review some basic definitions and facts on effect algebras. In Section 3, relationships between effect algebras with RDP and partially ordered Abelian groups with interpolation are discussed. In Section 4, we prove that any σ-orthocomplete atomic effect algebra with RDP is also an MV-effect algebra. Finally, in Section 5, we apply the results from Section 4 to a noncommutative generalization of effect algebras, called pseudo-effect algebras, to show when they are effect algebras and to describe the state space of such effect algebras.

2 Basic definitions and facts

Definition 2.1. [9] An *effect algebra* is a system $(E;+,0,1)$ consisting of a set E with two special elements 0 and 1, called the zero and the unit, and with a partially binary operation $+$ satisfying the following conditions for all $a,b,c \in E$:

(E1) If $a+b$ is defined, then $b+a$ is defined and $a+b=b+a$.

(E2) If $a+b$ is defined and $(a+b)+c$ is defined, then $b+c$ and $a+(b+c)$ are defined, and $(a+b)+c = a+(b+c)$.

(E3) For any $a \in E$, there exists a unique $b \in E$ such that $a+b$ is defined and $a+b=1$.

(E4) If $a+1$ is defined, then $a=0$.

Let a be an element of an effect algebra E and $n \geqslant 0$ be an integer. We define $na=0$ if $n=0$, $1a=a$ if $n=1$, and $na=(n-1)a+a$ if $(n-1)a$ and $(n-1)a+a$ are defined in E. We define the *isotropic index* $\imath(a)$ of the element a, as the maximal nonnegative number n such that na exists. If na exists for every integer n, we say that $\imath(a)=+\infty$.

Remark 2.2. [6] Let $(E;+,0,1)$ be an effect algebra.

(i) Define a partial binary relation $\leqslant$ on E by $a \leqslant b$ if, for some $c \in E$, we have $c+a=b$. Then $(E;\leqslant,0,1)$ is a poset, and $0 \leqslant a \leqslant 1$ for each $a \in E$. Furthermore, if $(E;\leqslant,0,1)$ is a lattice, then we say that $(E;+,0,1)$ is a lattice effect algebra.

(ii) Define a binary relation $\perp$ on E by $a \perp b$ if and only if $a+b$ exists in E.

(iii) Define a partial binary operation $-$ on E by $c-b=a$ if and only if $a+b=c$. Then the algebraic system $(E;-,0,1)$ is a D-poset, [6].

For a comprehensive review on effect algebras, see [6], where also unexplained notions from this paper can be found.

Let $(E;\leqslant)$ be a poset and let $a,b \in E$ be two elements such that $a \leqslant b$. Then we define an interval $E[a,b] := \{c \in E \mid a \leqslant c \leqslant b\}$.

We recall that a group $(G;+,0)$ written additively is a *partially ordered group* (po-group for short) if $a \leqslant b$ implies $c+a+d \leqslant c+b+d$ for all $c,d \in G$. If G with respect to $\leqslant$ is a lattice, we call G a lattice-ordered group (ℓ-group for short).

We denote by $G^+ := \{g \in G \mid 0 \leqslant g\}$ the positive cone of G. A po-group is *directed* if, for any $g_1, g_2 \in G$, there is an element $h \in G$ such that $g_1, g_2 \leq h$.

If G is a po-group and $u \in G^+$, then the interval $G^+[0,u]$ can be converted into an effect algebra if we say that, for $a,b \in G^+[0,u]$, $a+b$ is defined in $G^+[0,u]$ iff the group addition $a+b$ is in $G^+[0,u]$ and our addition $a+b$ coincides then with the group addition. Then $(G^+[0,u];+,0,u)$ is an effect algebra. Every effect algebra E which is isomorphic with some $G^+[0,u]$, where G is a po-group with strong unit u, is said to be an *interval effect algebra*.

Definition 2.3. (i) An element a of a poset E with the least element 0 is called an *atom*, if the interval $E[0,a] = \{x \in E \mid 0 \leqslant x \leqslant a\}$ equals the set $\{0,a\}$.

(ii) An effect algebra E is called *atomic* if, for any nonzero x of E, there exists an atom a in E such that $a \leqslant x$.

Definition 2.4. [6] An effect algebra $(E;+,0,1)$ has the *Riesz Decomposition Property* (RDP) if, for any a_1, a_2, b_1, $b_2 \in E$, the equality $a_1+a_2 = b_1+b_2$ implies the existence of four elements $c_{11}, c_{12}, c_{21}, c_2 \in E$ such that $a_i = c_{i1}+c_{i2}$, and $b_j = c_{1j}+c_{2j}$ for all $i,j \in \{1,2\}$.

We note that due to [6], an effect algebra $(E;+,0,1)$ has RDP iff, for a, b_1, $b_2 \in E$ with $a \leqslant b_1+b_2$, there exist a_1, $a_2 \in E$ such that $a = a_1+a_2$, and $a_i \leqslant b_i$ for all $i=1,2$.

Definition 2.5. [10] An Abelian po-group $(G;+,0)$ has the *Riesz Decomposition Property* (RDP) if, for any a, b_1, $b_2 \in G^+$ with $a \leqslant b_1+b_2$, there exist a_1, $a_2 \in G^+$ such that $a = a_1+a_2$, and $a_i \leqslant b_i$ for all $i \in \{1,2\}$.

3 Effect algebras with RDP and Abelian po-groups

Let M be a subset of a po-group G. We denote by $sss(M)$ the sub-semigroup of G consisting of all finite sums of elements M and of 0. An element $u \in G^+$ is said to be (i) a *strong unit* or an *order unit* if, given $g \in G$, there is an integer $n \geq 1$ such that $g \leqslant nu$, and (ii) a *generative unit* if $G^+ = sss(G^+[0,u])$ and $G = G^+ - G^-$. By [6, Lem 1.4.6], every generative unit is an order unit.

In this section, we give sufficient and necessary conditions such that a po-group G with a generative unit satisfies RDP.

Definition 3.1. Let E be an atomic effect algebra and $A(E)$ be the set of atoms of E.

(i) Two finite sequences of atoms in $A(E)$ $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are called *similar* if there exists a permutation $(p_1, \dots, p_n)$ of $(1, \dots, n)$ such that $a_i = b_{p_i}$, $i = 1, \dots, n$.

(ii) We say that E fulfils the *unique atom representable property* (UARP, for short) if, for any two finite sequences of atoms $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$ such that $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, then $m = n$ and the sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are similar.

Similarly, we can give the following definition for Abelian po-groups.

Definition 3.2. Let G be an Abelian po-group and let $A(G^+)$ be the set of atoms of G^+.

(i) Two finite sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ of atoms in $A(G^+)$ are called *similar* if there exists a permutation $(p_1, \dots, p_n)$ of $(1, \dots, n)$ such that $a_i = b_{p_i}$, $i = 1, \dots, n$.

(ii) We say that G fulfils the *unique atom representable property* (UARP, for short) if, for any two finite sequence of atoms $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$ in $A(G^+)$ such that $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, then $m = n$ and the sequences $(a_1, \dots, a_m)$ and $(b_1, \dots, b_m)$ are similar.

Proposition 3.3. *Let G be an Abelian po-group with a fixed element $u > 0$. If $G^+[0, u]$ satisfies the condition $G^+[0, u] + G^+[0, u] = G^+[0, 2u]$, then $A(G^+[0, 2u]) = A(G^+[0, u])$, where $A(G^+[0, u])$ and $A(G^+[0, 2u])$ refer to the sets of atoms of the effect algebras $G^+[0, u]$ and $G^+[0, 2u]$, respectively.*

Proof. Assume $a \in A(G^+[0, u])$, $b \in G^+[0, 2u]$ and $b < a$. Then $b < a \leqslant u$, so that $b \in G^+[0, u]$ and $a = 0$.

Conversely, assume that $a \in A(G^+[0, 2u])$. Then there exist two elements b, $c \in G^+[0, u]$ such that $a = b + c$, and so $b, c \leqslant a$. Since $a \in A(G^+[0, 2u])$, we have that either $b = 0$ or $c = 0$, and so $a \in G^+[0, u]$, which implies that $a \in A(G^+[0, u])$. $\square$

Proposition 3.4. *Let E be an effect algebra with RDP. Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$ be two finite sequences of atoms such that $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j$, then the sequences A and B are similar.*

Proof. If $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j$, by [7, Lem 3.9], there is a system $\{x_{ij} \mid i = 1, \dots, n,\ j = 1, \dots, m\}$ of elements from E such that

$$a_i = \sum_{j=1}^m x_{ij}, \quad b_j = \sum_{i=1}^n x_{ij}$$

for each $i = 1, \dots, n$ and each $j = 1, \dots, m$. Therefore, for any atom a_i there is a unique x_{ij_i} such that $a_i = x_{ij_i}$ and for any atom b_j there is a unique $x_{i_j j}$ such that $b_j = x_{i_j j}$. Hence, $n = m$ and the commutativity of $+$ entails A and B are similar. $\square$

Proposition 3.5. *Let G be a po-group with RDP and u be a generative unit, and let $E = G^+[0, u]$ be an atomic effect algebra. If, for any $x \in E$, there exists a finite sequence of atoms $a_1, \dots, a_n$ in E such that $x = a_1 + \cdots + a_n$, then the po-group G fulfils UARP.*

Proof. Firstly, the set $A(G^+) = \{a \mid a \text{ is atom of } G^+\}$ equals the set $A(E) = \{a \mid a \text{ is atom of } E\}$. Since E is atomic, we have $A(E) \neq \emptyset$. For any $a \in A(E)$, if $b \in G^+$ with $b < a$, then we have that $b < u$, which implies that $b = 0$, and so $a \in A(G^+)$. Conversely, if $a \in A(G^+)$, then $a \in G^+$, which implies that there exist $a_1, \ldots, a_n \in E$ such that $a = a_1 + \cdots + a_n$. Since a is an atom of G^+, we have that there exists a unique index $i \in \{1, \ldots, n\}$ such that $a = a_i$ and $a_j = 0$ with $j \neq i$. Hence, $a \in E$, thus $a \in A(E)$.

By $G^+ = ssg(E)$, for any $g \in G^+$, there exist $e_1, e_2, \ldots, e_s \in E$ such that $g = e_1 + e_2 + \cdots + e_s$. Furthermore, by the assumptions, for any $i \in \{1, 2, \ldots, s\}$, there exists a finite sequence of atoms $a_{i1}, a_{i2}, \ldots, a_{it_i} \in E$ such that $e_i = a_{i1} + a_{i2} + \cdots + a_{it_i}$, and so there exists a finite sequence of atoms $a_{11}, a_{12}, \ldots, a_{1t_1}, \ldots, a_{s1}, a_{s2}, \ldots, a_{st_s} \in E$ such that $g = a_{11} + a_{12} + \cdots + a_{1t_1} + \cdots + a_{s1} + a_{s2} + \cdots + a_{st_s}$. The rest part of the result follows the similar proof of Proposition 3.4. □

Theorem 3.6. *Let G be a po-group G fulfilling UARP and u be a generative unit. Then the following statements hold.*

(i) *$G^+[0, u]$ satisfies RDP.*

(ii) *For any natural $n \geqslant 1$, the effect algebra $G^+[0, nu]$ satisfies RDP.*

(iii) $G^+[0, nu] = \underbrace{G^+[0, u] + \cdots + G^+[0, u]}_{n-times}$.

(iv) *The po-group G satisfies RDP.*

Proof. (i) Assume that $x \leqslant y + z$ for $x, y, z \in G^+[0, u]$. Then there exists an element $w \in G^+[0, u]$ such that $x + w = y + z$. Since G satisfies UARP, there exist finite sequences of atoms $(x_1, \ldots, x_m)$, $(w_1, \ldots, w_q)$, $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_p)$ such that $x = x_1 + \cdots + x_m$, $w = w_1 + \cdots + w_q$, $y = y_1 + \cdots + y_n$ and $z = z_1 + \cdots + z_p$, and so $x_1 + \cdots + x_m + w_1 + \cdots + w_q = y_1 + \cdots + y_m + z_1 + \cdots + z_p$. Hence, the sequences $(x_1, \ldots, x_m, w_1, \ldots, w_q)$ and $(y_1, \ldots, y_n, z_1, \ldots, z_p)$ are similar, thus for any $i \in \{1, 2, \ldots, m\}$ there exists a unique $y_{p(i)}$ or a unique $z_{q(i)}$ such that $x_i = y_{p(i)}$ or $z_{q(i)}$. Set $I_1 = \{i \mid \text{there exists } y_{p(i)} \text{ such that } x_i = y_{p(i)}\}$, $I_2 = \{i \mid \text{there exists } z_{q(i)} \text{ such that } x_i = z_{q(i)}\}$ and we get $a = \sum_{i \in I_1} y_{p(i)}$, $b = \sum_{i \in I_2 \setminus I_1} z_{q(i)}$. Thus, we have that $x = a + b$ and $a \leqslant y$, $b \leqslant z$.

(ii) In any rate, $G^+[0, u] \subseteq G^+[0, nu]$, and so $G^+ = ssg(G^+[0, nu])$ which yields that also nu is a generative unit. By (i), we have that the effect algebra $G^+[0, nu]$ satisfies RDP.

(iii) It is easy to see that $G^+[0, nu] \supseteq \underbrace{G^+[0, u] + \cdots + G^+[0, u]}_{n-times}$. By (ii), the effect algebra $G^+[0, nu]$ satisfies RDP, and so, for any $x \in G^+[0, nu]$, there exist n elements $x_1, x_2, \ldots, x_n$ such that $x = x_1 + x_2 + \cdots + x_n$, which implies $x \in \underbrace{G^+[0, u] + \cdots + G^+[0, u]}_{n-times}$.

(iv) For any $a, b, c, d \in G^+$, if $a + b = c + d$, then there exists a natural number n such that $a + b \leqslant nu$. By (ii), $G^+[0, nu]$ satisfies RDP, which implies there exist x_1, x_2, x_3, $x_4 \in G^+[0, nu]$, such that $a = x_1 + x_2$, $b = x_3 + x_4$, $c = x_1 + x_3$, $d = x_2 + x_4$. □

An easy corollary of Theorem 3.6 is the following result.

Corollary 3.7. *Let G be a po-group with a generative unit u and let $E = G^+[0,u]$ If, for any $x \in E$, there exists a finite sequence of atoms $a_1, \dots, a_n$ in E such that $x = a_1 + \cdots + a_n$. Then G satisfies RDP if and only if G satisfies UARP.*

We say that a poset E satisfies the *Riesz Interpolation Property* (RIP), or G is with *interpolation*, if $a_1, a_2 \leqslant b_1, b_2$, then there is an element $c \in E$ such that $a_1, a_2 \leqslant c \leqslant b_1, b_2$. Then an Abelian po-group G satisfies RIP iff G satisfies RDP, iff G^+ satisfies the same property as an effect algebra with RDP, see [10, Prop 2.1].

Example 3.8. [6] Let G be the Abelian group $\mathbb{Z}^2$ with the positive cone $G^+ = \{(a,b) \in G | 2a \geqslant b \geqslant 0\}$.

(i) G does not fulfill RIP.

Set $x_1 = (0,0)$ and $x_2 = (0,1)$, while $y_1 = (1,1)$ and $y_2 = (1,2)$. Then $x_i \leqslant y_j$ for all i, j, but there is no element $z \in G$ such that $x_i \leqslant z \leqslant y_j$ for all i, j.

(ii) The element $u = (2,1)$ is a strong unit of G.

For any $(a,b) \in G$, there exists positive element m such that $(a,b) \leqslant m(2,1) = (2m, m)$. Notice that $(a,b) \leqslant n(2,1) = (2n, n)$ iff $4n - 2a \geqslant n - b \geqslant 0$ iff $3n \geqslant 2a - b$, $n \geqslant b$. Let $n_0 = \max\{1, b, [\frac{1}{3}(2a - b)] + 1\}$. Now, we set $m = n_0$, then we have that $m \geqslant 1$, and so $2m \geqslant m = \max\{1, b, [\frac{1}{3}(2a - b)] + 1\}$, hence, the inequality $(a,b) \leqslant m(2,1) = (2m, m)$ holds.

For any $(a,b), (c,d) \in G$, there exist two positive integers m_1, m_2 such that $(a,b) \leqslant m_1(2,1)$, $(c,d) \leqslant m_2(2,1)$. Set $m = \max\{m_1, m_2\}$, we have that $(a,b), (c,d) \leqslant m(2,1)$. Hence, we have prove that G is directed and the positive element $(2,1)$ is a strong unit.

(iii) By (ii), the po-group G is directed and we have $G = G^+ - G^+$.

(iv) Let 0 and u denote the elements $(0,0)$ and $(2,1)$, respectively. Then the set $G^+[0,u] = \{0, (1,0), (1,1), u\}$ is an interval effect algebra satisfying RDP.

(v) Observe that $G^+ = \bigcup_{n \in \mathbb{N}} G^+[0, nu]$ and $G^+ \neq ssg(G^+[0,u])$, and so u is not a generative strong unit for G. The po-group G is not an ambient group with order unit u for E (we say that a po-group G with a generative unit u is *ambient* for an effect algebra E if E is isomorphic to $G^+[0,u]$).

$G^+[0,2u] = \{0, (1,0), (2,0), (3,0), (1,1), (2,1), (3,1), (1,2), (2,2), (3,2), (4,2)\}$,

$G^+[0,2u] \neq G^+[0,u] + G^+[0,u]$. Notice that $(1,2) \in G^+[0,2u] \subseteq G^+$, but for any natural number $n \geqslant 1$, there exist no elements $x_i \in G^+[0,u]$, $i = 1, \dots, n$, such that $(1,2) = x_1 + \cdots + x_n$, that is $(1,2) \notin ssg(G^+[0,u])$.

(vi) Although $G^+[0,u]$ is a Boolean algebra, the effect algebra $G^+[0,2u]$ does not satisfy neither RDP nor RIP.

For example, $(3,0) + (1,2) = (3,1) + (1,1)$, however, there do not exist any elements $x_1, x_2, x_3, x_4 \in G^+[0,2u]$ such that $(3,0) = x_1 + x_2$, $(1,2) = x_3 + x_4$ and $(3,1) = x_1 + x_3$, $(1,1) = x_2 + x_4$.

For $(2,0), (2,1) \leqslant (3,1), (3,2)$, there exists no element $x \in G^+[0,2u]$ such that $(2,0), (2,1) \leqslant x \leqslant (3,1), (3,2)$.

Example 3.9. Let G be the Abelian group $\mathbb{Z}$, and G^+ be the set $\{n \in \mathbb{Z} \mid n = 0, \text{ or } n \geqslant 2\}$. Then G^+ is a strict cone, and so G is a po-group with the partially order $\leqslant_1$, for any $a, b \in G$, $a \leqslant_1 b$ iff

$b - a \in G^+$. Let $u = 5$, then it is easy to see that the positive element u is a strong unit of G and $G^+ = ssg(E)$, where $E = G^+[0, 5]$.

The equation $G^+[0, nu] = \underbrace{G^+[0, u] + \cdots + G^+[0, u]}_{n-times}$ holds for any natural number n.

The interval effect algebra $G^+[0, 5]$ is isomorphic to the Boolean algebra 2^2 which satisfies RDP. $G^+[0, 10]$ does not satisfies RDP. In fact, $3 + 3 = 2 + 4$, however, there exist no elements x_1, x_2, x_3, $x_4 \in G^+[0, 10]$ such that $3 = x_1 + x_2$, $3 = x_3 + x_4$, $2 = x_1 + x_3$, $4 = x_2 + x_4$.

For any natural number $n \geqslant 2$, the effect algebra $G^+[0, 5n]$ does not fulfil RIP. In fact, $3, 4, 6, 7 \in G^+[0, 5n]$, with $3, 4 \leqslant_1 6, 7$, however, there is no element $i \in G^+[0, 5n]$ such that $3, 4 \leqslant_1 i \leqslant_1 6, 7$.

Remark 3.10. (i) Let G be a po-group with the positive cone G^+, and let a positive element u be a strong unit. Then the equation $G^+ = ssg(G^+[0, u])$ does not hold in general. See Example 3.8.

(ii) Let G be a po-group with the positive cone G^+ and let a positive element u be a strong unit. Then the equation $G^+[0, nu] = \underbrace{G^+[0, u] + \cdots + G^+[0, u]}_{n-times}$ does not hold, in general. See the Example 3.8.

(iii) Let G be a po-group with the positive cone G^+, and a positive element u be a strong unit. Assume that the positive cone $G^+ = ssg(G^+[0, u])$ and $G^+[0, nu] = \underbrace{G^+[0, u] + \cdots + G^+[0, u]}_{n-times}$ for any natural number $n \geqslant 1$. However, Example 3.9 shows that although the effect algebra $G^+[0, u]$ satisfy the RDP, the effect algebra $G^+[0, 2u]$ does not fulfils RDP, which implies that po-group G does not fulfils RDP.

4 Orthocomplete atomic effect algebra with RDP

In the present section, we show that every orthocomplete atomic effect algebra with RDP is an MV-effect algebra.

We recall that two elements a and b of an effect algebra E are *compatible*, if there exist three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c$ is defined in E. We say that a lattice-ordered effect algebra E is an *MV-effect algebra* if all elements of E are mutually compatible.

It is known that if a lattice-ordered effect algebra E satisfies RDP, then it is also an MV-effect algebra [16].

Now, we prove that chain finite effect algebras with RDP are MV-algebras. Firstly, we recall some useful results for effect algebras with RDP.

Lemma 4.1. [1] *Let E be an effect algebra with RDP. If E is a finite set, then E is an MV-effect algebra.*

Definition 4.2. [6] *Let E be an effect algebra. If every chain in E is a finite set, then we say that E satisfies the chain condition.*

Lemma 4.3. [9] *If an effect algebra E satisfies the chain condition, then every nonzero element in E is a finite orthogonal sum of atoms.*

Theorem 4.4. *If an effect algebra E with RDP satisfies the chain condition, then*

(i) *E is a finite set.*

(ii) *E is an MV-effect algebra.*

Proof. By Lemma 4.1, it suffices to prove that the statement (i) holds. Since E satisfies the chain condition, then there exists a finite sequence of atoms $A = (x_1, x_2, \dots, x_n)$ such that $1 = x_1 + x_2 + \cdots + x_n$. By Proposition 3.4, for any other sequence of atoms $B = (b_1, b_2, \dots, b_m)$ such that $1 = b_1 + b_2 + \cdots + b_m$, we have that these two sequences of atoms A and B are similar. Now, for any atom a, we have that $a + a' = 1$. There exists a sequence of atoms $C = (c_1, c_2, \dots, c_m)$ such that $a' = c_1 + c_2 + \cdots + c_m$, which implies that the sequence $(a, c_1, c_2, \dots, c_m)$ is similar to the sequence $A = (x_1, x_2, \dots, x_n)$. Hence, $a \in \{x_i \mid i = 1, \dots, n\}$. Thus, the set of atoms of E equals $\{x_i \mid i = 1, \dots, n\}$. Therefore, for any $x \in E$ with $x \neq 0$, $x \leqslant 1 = x_1 + x_2 + \cdots + x_n$. By RDP, there exists a finite sequence of atoms y_1, y_2, $\dots$, y_m with $m \leqslant n$ such that $x = y_1 + y_2 + \cdots + y_m$, where $y_j \in \{x_i \mid i = 1, \dots, n\}$ for any $j = 1, 2, \dots, m$. Hence, there exists at most 2^n elements in E, which implies that E is a finite set. $\square$

In general, effect algebras with the chain condition but without RDP are not necessarily finite as the following example shows.

Example 4.5. Assume that E is the horizontal sum of a system $(E_i)_{i \in \mathbb{N}}$ of effect algebras, where $E_i = \{0, a_i, 1\}$ is a three-element chain effect algebra for any $i \in \mathbb{N}$. Then E is a chain finite atomic effect algebra without RDP, and it is also infinite.

In the following, we prove that atomic σ-orthocomplete effect algebras with RDP are also MV-effect algebras.

Let E be an effect algebra. We say that a finite sequence $F := (a_1, a_2, \dots, a_n)$ is *orthogonal* if $a_1 + a_2 + \cdots + a_n$ exists in E, and then we write $a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$, and the element $\sum_{i=1}^{n} a_i$ is called the *sum* of the finite system F. The sum of the system F is written as $\sum F$.

For an arbitrary system $A = (a_i)_{i \in I}$ of not necessarily different elements of E, we say that G is *orthogonal*, if every finite subsystem F of A is orthogonal. Furthermore, for an arbitrary orthogonal system A, if the supremum $\bigvee\{\sum F \mid F$ is a finite subsystem of $A\}$ exists in E, then we say that the element $\bigvee\{\sum F \mid F$ is a finite subsystem of $A\}$ is the *sum* of A. The sum of the system A is written as $\sum A$.

We say that an effect algebra is *orthocomplete* if an arbitrary orthogonal system has a sum. Especially, we say that an effect algebra is *σ-orthocomplete* if every countable orthogonal system has a sum.

Remark 4.6. By [12, Thm 3.2], an effect algebra E is σ-orthocomplete iff, for any countable increasing chain $(a_i)_{i\in\mathbb{N}}$, the supremum $\bigvee_{i\in\mathbb{N}} a_i$ exists in E.

Theorem 4.7. *Let E be a σ-orthocomplete atomic effect algebra with RDP and $A(E)=\{a_i \mid i\in\mathbb{N}\}$ be the set of all atoms of E. Then the following statements hold.*

(i) *For any $a_i, a_j \in A(E)$ with $a_i \neq a_j$, then $a_i + a_j$ and $a_i \vee a_j$ exists and $a_i + a_j = a_i \vee a_j$.*

(ii) *For any natural number $n \geqslant 2$, the finite set of mutually different atoms $\{a_1,\ldots,a_n\} \subseteq A(E)$ is orthogonal in E and $\sum_{i=1}^n a_i = \bigvee_{i=1}^n a_i$.*

(iii) *The system $A(E)$ is an orthogonal system, and $\sum A(E) = \bigvee A(E)$.*

Proof. (i) See the Lemma 3.2 (ii) in [7].

(ii) We will proceed by mathematical induction with respect to n.

For $n=2$, by (i), $\{a_1,a_2\}$ is orthogonal and $a_1+a_2=a_1\vee a_2$.

Assume that the statement holds for any $m' < m$. For a finite set of mutually different atoms $\{a_1,\ldots,a_m\}$, by induction hypothesis, we have that $\sum_{i=1}^{m-1} a_i = \bigvee_{i=1}^{m-1} a_i$. Noticing that for any $i \in \{1,\ldots,m-1\}$, a_i+a_m exists, and so $(\bigvee_{i=1}^{m-1} a_i)+a_m$ exists, which implies that the sum $\sum_{i=1}^m a_i$ exists. Now it suffices to prove that $\sum_{i=1}^m a_i = \bigvee_{i=1}^m a_i$.

Assertion: If $x \leqslant a_m$ and $x \leqslant \bigvee_{i=1}^{m-1} a_i$, then $x=0$.

Since a_m is an atom and if $x \leqslant a_m$, then $x=0$ or $x=a_m$. Assume that $x=a_m$, then there exists an element b_1 such that $a_m+b_1 = (\sum_{i=1}^{m-2} a_i)+a_{m-1}$. By RDP and $a_m \wedge a_{m-1}=0$, we will get that $a_m \leqslant \sum_{i=1}^{m-2} a_i$. Then there exists an element b_2 such that $a_m+b_2 = (\sum_{i=1}^{m-3} a_i)+a_{m-2}$. Repeating the same process, we will find an element b such that $a_m+b=a_1+a_2$, by RDP, we get that $a_m \leqslant a_1$ or $a_n \leqslant a_2$, which is a contradiction. Consequently, $x=0$.

Now, we assume that $a_m, \bigvee_{i=1}^{m-1} a_i \leqslant x$. Then there exist $a,b\in E$ such that $a_m+a = (\bigvee_{i=1}^{m-1} a_i)+b = x$. Using RDP and the assertion, we have that $a_m \leqslant b$, which implies that $\sum_{i=1}^m a_i = (\bigvee_{i=1}^{m-1} a_i)+a_m \leqslant u$. Hence, $\sum_{i=1}^m a_i = \bigvee_{i=1}^m a_i$.

(iii) By (ii), the system $A(E)=\{a_i \mid i\in\mathbb{N}\}$ is an orthogonal system. Since E is a σ-orthocomplete effect algebra, we have that $\sum A(E) = \bigvee\{\sum F \mid F$ is a finite subset of $A(E)\} = \bigvee\{\sum_{i=1}^n a_i \mid n\in\mathbb{N}, n\geqslant 1\} = \bigvee_{i\in\mathbb{N}} a_i$. □

The following result generalizes an analogous result from [5].

Theorem 4.8. *Let E be a σ-orthocomplete atomic effect algebra with RDP and $A(E)=\{a_i \mid i\in I\}$ be the set of all atoms of E which is at most countable. Let $\imath_i$ be the isotropic index of $a_i\in A(E)$. The following statements hold.*

(i) *For any $a_i \in A(E)$, the isotropic index $\imath_i$ is finite, $i\in I$.*

(ii) *For any $a_i\in A(E)$, the interval $E[0,\imath_i a_i] = \{x\in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals to $\{0,a_i,\ldots,\imath_i a_i\}$.*

(iii) *For any two distinct elements $a_i,a_j\in A(E)$, $(\imath_i a_i)\wedge(\imath_j a_j)$ exists and $(\imath_i a_i)\wedge(\imath_j a_j)=0$.*

(iv) *For any two distinct elements* $a_i, a_j \in A(E)$, $(\imath_i a_i) + (\imath_j a_j)$ *exists and* $(\imath_i a_i) + (\imath_j a_j) = (\imath_i a_i) \vee (\imath_j a_j)$.

(v) *The system* $\{\imath_i a_i \mid a_i \in A(E)\}$ *is an orthogonal system, and* $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

Proof. (i) For any $a_i \in A(E)$, if the sum na_i exists for any natural number $n \geq 1$, then we get an infinite chain $a_i < 2a_i < \cdots < na_i < \cdots < 1$. Since the effect algebra E is σ-orthocomplete, then $\bigvee_n na_i$ exists. Let $x = \bigvee_n na_i$, then $x = \bigvee_n (n+1)a_i = a_i + (\bigvee_n na_i) = a_i + x$ which implies that $a_i = 0$. This is a contradiction with the definition of a_i. Hence, the isotropic $\imath_i$ of a_i is finite.

(ii) For any $x \in E[0, \imath_i a_i]$, if $x = 0$ or $x = \imath_i a_i$, then the result holds. Now, if $0 < x < \imath_i a_i$, then there exists $y \in E$ such that $x + y = \imath_i a_i$. By RDP, there exist $x_{11}, \ldots, x_{1i} \in E$ and $x_{21}, \ldots, x_{2i} \in E$ such that $a_i = x_{11} + x_{21} = \cdots = x_{1i} + x_{2i}$, and $x = x_{11} + \cdots + x_{1i}$, $y = x_{21} + \cdots + x_{2i}$. Since a_i is an atom of E, we have that $x_{11}, \ldots, x_{1i}, x_{21}, \ldots, x_{2i} \in \{0, a_i\}$, which implies that there exists a natural number $1 \leqslant n \leqslant \imath_i$, such that $x = na_i$.

(iii) For any $x \in E$ with $x \leqslant \imath_i a_i, \imath_j a_j$, we have that $x \in \{0, a_i, \ldots, \imath_i a_i\} \cap \{0, a_j, \ldots, \imath_j a_j\} = 0$, which implies that $(\imath_i a_i) \wedge (\imath_j a_j)$ exists and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$.

(iv) Without loss of generality, we just prove that $(\imath_1 a_1) + (\imath_2 a_2)$ exists and $(\imath_1 a_1) + (\imath_2 a_2) = (\imath_1 a_1) \vee (\imath_2 a_2)$.

Noticing that $(\imath_1 a_1) + (\imath_1 a_1)' = (\imath_2 a_2) + (\imath_2 a_2)'$, by RDP, there exist x_1, x_2, x_3, $x_4 \in E$ such that $\imath_1 a_1 = x_1 + x_2$, $(\imath_1 a_1)' = x_3 + x_4$, $\imath_2 a_2 = x_1 + x_3$, $(\imath_2 a_2)' = x_2 + x_4$, which implies $x_1 = 0$ by (iii). Hence, $\imath_1 a_1 = x_2 \leqslant (\imath_2 a_2)'$, which implies that $(\imath_1 a_1) + (\imath_2 a_2)$ exists in E. Furthermore, assume that $\imath_1 a_1$, $\imath_2 a_2 \leqslant u$, for $u \in E$. Then there exist u_1, $u_2 \in E$ such that $(\imath_1 a_1) + u_1 = (\imath_2 a_2) + u_2$, again by (iii) and RDP, we get that $\imath_1 a_1 \leqslant u_2$, which implies that $(\imath_1 a_1) + (\imath_2 a_2) \leqslant u$. Hence, we have that $(\imath_1 a_1) + (\imath_2 a_2) = (\imath_1 a_1) \vee (\imath_2 a_2)$.

(v) By (iv), the system $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\}$. Now, obviously $\sum\{\imath_i a_i \mid a_i \in A(E)\} \leqslant 1$. If $\sum\{\imath_i a_i \mid a_i \in A(E)\} < 1$, then there exists an element $x \in E$ such that $(\sum\{\imath_i a_i \mid a_i \in A(E)\}) + x = 1$. Since E is atomic, there exists an atom $a_x \leqslant x$. Hence, we have that $(\sum\{\imath_i a_i \mid a_i \in A(E)\}) + a_x$ exists in E. But $a_x \in A(E)$, we have that $a_x + (\imath_{a_x} a_x)$ exists in E, which is a contradiction. Consequently, we have that $\sum\{\imath_i a_i \mid a_i \in A(E)\} = 1$. □

We recall that *central elements* of an effect algebra were defined in [6, Def 1.9.11]. If $C(E)$ is the set of central elements, then $C(E)$ is a Boolean algebra. If E satisfies RDP, then an element $e \in E$ is central iff $e \wedge e' = 0$, see [4, Thm 3.2].

Lemma 4.9. [12] *Let E be an orthocomplete effect algebra. Let $(a_\alpha : \alpha \in \Sigma) \subseteq E$ be an orthogonal family of central elements. Let $(x_\alpha : \alpha \in \Sigma)$ be a family of elements satisfying $x_\alpha \leqslant a_\alpha$, for all $\alpha \in \Sigma$. Then $\bigvee(x_\alpha : \alpha \in \Sigma)$ exists and equals $\sum(x_\alpha : \alpha \in \Sigma)$.*

Lemma 4.10. [12] *Let E be an orthocomplete effect algebra. Let $(a_\alpha \mid \alpha \in \Sigma)$ be an orthogonal family of central elements. Denote $a = \sum(a_\alpha \mid \alpha \in \Sigma)$. Then the element a is central and $E[0,a]$ is isomorphic to the product $\prod_{\alpha\in\Sigma} E[0,a_\alpha]$.*

Theorem 4.11. *Let E be a σ-orthocomplete atomic effect algebra with RDP and $A(E) = \{a_i \mid i \in I\}$ be the set of all atoms of E which at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. Then the following statements hold.*

(i) *For any $a_i \in A(E)$, the element $\imath_i a_i$ is a central elements.*

(ii) *For any $a_i \in A(E)$, the element $\imath_i a_i$ is an atom of the Boolean algebra $C(E)$.*

(iii) *For any $y \in E$, $y = \sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\}$.*

(iv) *The effect algebra E is isomorphic to the product effect algebra $\prod_{i\in I} E[0,\imath_i a_i]$.*

(v) *The effect algebra E is a σ-complete MV-effect algebra.*

Proof. (i) By [4, Thm 3.2], it suffices to prove that $\imath_i a_i \wedge (\imath_i a_i)' = 0$. Assume that $x \leqslant \imath_i a_i, (\imath_i a_i)'$. If $x \neq 0$, then $a_i \leqslant (\imath_i a_i)'$ by Theorem 4.8 (ii), and so $a_i + (\imath_i a_i)$ exists, which is a contradiction. Thus, $x = 0$, and so $\imath_i a_i \wedge (\imath_i a_i)' = 0$.

(ii) For any $x \in E$, $x < \imath_i a_i$, we have that $x \in \{0, a_i, \ldots, (\imath_i - 1)a_i\}$ by Theorem 4.8 (ii). If $x \neq 0$, then $a_i \leqslant x, x'$ by Theorem 4.8 (ii), which implies that $x \notin C(E)$. Hence, we have that $\imath_i a_i$ is an atom of $C(E)$.

(iii) Since $\imath_i a_i \in C(E)$, for any $y \in E$, $y \wedge \imath_i a_i$ exists in E. Since the set $\{\imath_i a_i \mid a_i \in A(E)\}$ is orthogonal, the set $\{y \wedge \imath_i a_i \mid a_i \in A(E)\}$ is also orthogonal and so the sum $\sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\}$ exists in E. Notice that for any two elements $y \wedge \imath_i a_i$, $y \wedge \imath_j a_j \in \{y \wedge \imath_i a_i \mid a_i \in A(E)\}$, the sum $(y \wedge \imath_i a_i) + (y \wedge \imath_j a_j)$ exists and $(y \wedge \imath_i a_i) + (y \wedge \imath_j a_j) = (y \wedge \imath_i a_i) \vee (y \wedge \imath_j a_j)$ by Lemma 4.9. Whence, for any finite subset $F \subseteq \{y \wedge \imath_i a_i \mid a_i \in A(E)\}$, we have that $\sum\{x \mid x \in F\} = \bigvee\{x \mid x \in F\}$. In addition, we have that $\sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\} = \bigvee\{y \wedge \imath_i a_i \mid a_i \in A(E)\} \leqslant y$. Assume that $\sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\} = \bigvee\{y \wedge \imath_i a_i \mid a_i \in A(E)\} < y$. Then there exists an element $x \in E$ such that $x + (\sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\}) = y$, and so there exists an atom $a_{i_0} \in E$ such that $a_{i_0} \leqslant x$. However, $x + (y \wedge \imath_{i_0} a_{i_0})$ exists, and so $a_{i_0} + (y \wedge \imath_{i_0} a_{i_0}) \leqslant y, \imath_{i_0} a_{i_0}$, hence, $a_{i_0} + (y \wedge \imath_{i_0} a_{i_0}) \leqslant y \wedge (\imath_{i_0} a_{i_0})$, which is a contradiction. Thus, we have that $\sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\} = \bigvee\{y \wedge \imath_i a_i \mid a_i \in A(E)\} = y$.

(iv) By Theorem 4.8 (v) and Lemma 4.10, the statement holds.

(v) For any $i \in I$, the chain $E[0,\imath_i a_i]$ is a finite MV-effect algebra, and so it is complete. Hence, the product $\prod_{i\in I} E[0,\imath_i a_i]$ is also a σ-complete MV-effect algebra. □

Remark 4.12. In Proposition 3.12 of [5], the authors proved that any atomic σ-complete Boolean D-poset with the countable set of atoms $\{a_i \mid i \in I\}$ can be expressed as a direct product of finite chains. In fact, any Boolean D-poset is an MV-effect algebra, which is also a lattice-ordered effect algebra with RDP [6]. By Theorem 4.11, we can see that any σ-orthocomplete atomic effect algebra with RDP is also a lattice-ordered, thus it is an MV-algebra. Furthermore, similar to Theorem 4.11, we can prove the following result.

Theorem 4.13. *Let E be an orthocomplete atomic effect algebra with RDP and $A(E) = \{a_i \mid i \in I\}$ be the set of atoms of E. Then the following statements hold.*

(i) *For any $a_i \in A(E)$, the element $\imath_i a_i$ is a central elements.*

(ii) *For any $a_i \in A(E)$, the element $\imath_i a_i$ is an atom of Boolean algebra $C(E)$.*

(iii) *For any $y \in E$, $y = \sum\{y \wedge \imath_i a_i \mid a_i \in A(E)\}$.*

(iv) *The effect algebra E is isomorphic to the product effect algebra $\prod_{i\in I} E[0, \imath_i a_i]$.*

(v) *The effect algebra E is a complete MV-effect algebra.*

5 Applications

In the present section, we apply the methods and the results of the previous sections to a noncommutative generalization of effect algebras, pseudo-effect algebras, and to a description of the state space of some effect algebras.

A noncommutative generalization of effect algebras was introduced in [7, 8] and some additional basic properties can be found in [4].

Definition 5.1. [7] A structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra*, if for all $a, b, c \in E$, the following hold.

(PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.

(PE2) There are exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.

(PE3) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.

(PE4) If $a + 1$ or $1 + a$ exists, then $a = 0$.

We recall that a pseudo-effect algebra E is an effect algebra iff the partial addition $+$ is commutative, i.e. $a + b$ exits in E iff $b + a$ is defined in E, and the $a + b = b + a$.

In the same way as for effect algebras, we define for pseudo-effect algebra (i) the isotropic index, $\imath(a)$, of any element a of a pseudo-effect algebra, (ii) atom, (iii) atomic system, (iv) RDP, (v) central element, and (vi) center $C(E)$.

We say that a pseudo-effect algebra E is *monotone σ-complete* provided that every ascending sequence $x_1 \leqslant x_2 \leqslant \cdots$ of elements in E has a supremum $x = \bigvee_n x_n$. We recall that if E is an effect algebra, then the notions σ-orthocomplete effect algebras and monotone σ-complete effect algebras are equivalent. For pseudo-effect algebras, the notion of the σ-orthocomplete pseudo-effect algebra is not straightforward in view of the non-commutativity of the partial addition $+$. Therefore, for our aims, we prefer the notion of the monotone σ-completeness of pseudo-effect algebras.

Theorem 5.2. *Let E be a monotone σ-complete atomic pseudo-effect algebra with RDP and $A(E) = \{a_i \mid i \in I\}$ be the set of atoms of E that is at most countable. Then E is a commutative PEA, i.e., E is an effect algebra.*

Proof. For any two atoms a, b with $a \neq b$, we have $a \wedge b = 0$, so that by [4, Lem 3.1], $a + b$, $b + a$ and $a \vee b$ exists in E and they are equal.

We assert that the isotropic index of any atom a of E is finite. Indeed, assume the converse, i.e. $\imath(a) = \infty$. Then $na \in E$ for any integer $n \geqslant 1$, and $\bigvee_n na \in E$. Hence, $\bigvee_n (n+1)a = \bigvee_n na + a$ implies the contradiction $a = 0$.

In the same way as in Theorem 4.8(i) we prove that every $\imath(a)a$ is a central element.

Since $A(E)$ is at most countable, we assume that $A(E) = \{a_1, a_2, \ldots\}$. The RDP implies that, for all $a, b \in A(E)$, $\imath(a)a \wedge \imath(b)b = 0$, which yields that $\imath(a)a + \imath(b)b = \imath(b)b + \imath(a)a = \imath(a)a \vee \imath(b)b$. In the same way, we can show that if $a_1, \ldots, a_n$ are mutually different atoms, then $\imath(a_1)a_1 + \cdots + \imath(a_n)a_n$ exists in E and it equals $b_n := \imath(a_1)a_1 \vee \cdots \vee \imath(a_n)a_n$. In addition, $b_n = \imath(a_{j_1})a_{j_1} + \cdots + \imath(a_{j_n})a_{j_n}$ for any permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$. Thus, we have that $\{b_n\}$ is an ascending sequence and so $\bigvee_n b_n$ exists in E and we claim that $\bigvee_n b_n = 1$. In fact, if $\bigvee_n b_n < 1$, then there exists an atom a such that $\bigvee_n b_n + a \leqslant 1$, and so, $\imath(a)a + a$ exists in E which is absurd. Hence, $\bigvee_n \imath(a_n)a_n = \bigvee_n b_n = 1$. By [4, Thm 5.11], we have $x = \bigvee_n (x \wedge \imath(a_n)a_n)$ for any $x \in E$. Therefore, by [4, Pro 6.1(ii)], there exists an isomorphism $\phi : E \to \prod_{i\in I} E[0, \imath(a_i)a_i]$, where $\phi(x) = (x \wedge \imath(a_i)a_i)_{i\in I}$. Further, for any $x \in E$ and any $i \in \mathbb{N}$, we have $x \wedge \imath(a_i)a_i \in E[0, \imath(a_i)a_i] = \{0, a, \ldots, \imath(a_i)a_i\}$ by RDP.

For any $x, y \in E$, $x + y$ exists in E iff $(x \wedge \imath(a_i)a_i) + (y \wedge \imath(a_i)a_i)$ exists. Thus, if $x + y$ exists in E, then we have that for any $i \in I$, $(x+y)\wedge\imath(a_i)a_i = (x\wedge\imath(a_i)a_i)+(y\wedge\imath(a_i)a_i) = (y\wedge\imath(a_i)a_i)+(x\wedge\imath(a_i)a_i)$ which implies that $y + x$ exists and $x + y = y + x$. $\square$

Now we apply Theorem 4.13 for the description of states on some atomic effect algebras.

We say that a *state* on an effect algebra E is a mapping $s : E \to [0, 1]$ such that (i) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in E, and (ii) $s(1) = 1$. A state is an analogue of a probability measure. A state s is said to be *extremal* if, for any states s_1, s_2 and $\alpha \in (0, 1)$, the equation $s = \alpha s_1 + (1-\alpha)s_2$ implies $s = s_1 = s_2$. Let $\mathcal{S}(E)$ and $\partial_e\mathcal{S}(E)$ denote the set of all states and extremal states, respectively, on E. We recall that it can happen that an effect algebra is stateless. But every interval effect algebra admits at least one state, see [10, Cor 4.4]. We say that a net of states $\{s_\alpha\}_\alpha$ *converges weakly* to a state s iff $\lim_\alpha s_\alpha(a) = s(a)$ for any $a \in E$. Then $\mathcal{S}(E)$ is a compact Hausdorff space, and due to the Krein–Mil'man Theorem, see e.g. [10, Thm 5.17], every state is a weak limit of a net of convex combinations of extremal states on E.

We recall that a state on an MV-effect algebra is extremal, [14], iff $s(a \wedge b) = \max\{s(a), s(b)\}$ for all $a, b \in E$.

A state s is σ-additive if for any monotone sequence $\{a_i\}$ such that $\bigvee_i a_i = a$ implies $s(a) = \lim_i s(a_i)$. Equivalently, if $a = \sum_n a_n$, then $s(a) = \sum_n s(a_n)$.

Theorem 5.3. *Let E be a σ-orthocomplete atomic effect algebra with RDP and $A(E) = \{a_i \mid i \in I\}$ be the set of all atoms of E that is at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. For any $i \in I$, we define a mapping $s_i : E \to [0,1]$ via*

$$s_i(a) = \max\{j \mid ja_i \leqslant a \wedge \imath_i a_i\}/\imath_i, \quad a \in E.$$

Then s_i is an extremal state on E which is also σ-additive. If s is a σ-additive state on E, then $s(a) = \sum_i \lambda_i s_i(a)$, $a \in E$. Moreover, every extremal state that is also σ-additive is just of the form s_i for a unique i, and a state $s = s_i$ for some $i \in I$ if and only if $s(\imath_i a_i) = 1$.

Proof. By Theorem 4.13(i),(iii), the element $\imath_i a_i$ is central and $a = \sum_i \{a \wedge \imath_i a_i\}$. Therefore, $s_i(a)$ is a real number from the real interval $[0,1]$ and $(a+b) \wedge \imath_i a_i = (a \wedge \imath_i a_i) + (b \wedge \imath_i a_i)$ which proves that s_i is a state. Since by Theorem 4.13(v), E is an MV-effect algebra. If $a, b \in E$, we have $(a \wedge b) \wedge \imath_i a_i) = (a \wedge \imath_i a_i) \wedge (b \wedge \imath_i a_i)$ which implies $s_i(a \wedge b) = \min\{s_i(a), s_i(b)\}$ which proves s_i is an extremal state.

By [4, Thm 5.11], if $x_n \nearrow x$ and e is a central element, then $(\bigvee_n x_n) \wedge e = \bigvee_n (x_n \wedge e)$. From this and the definition of s_i, we have that each s_i is σ-additive.

Let s be an arbitrary σ-additive state, then $a = \sum_i \{a \wedge \imath_i a_i\}$ and $1 = \sum_i \imath_i a_i$, so that $s(a) = \sum_i s(a \wedge \imath_i a_i) = \sum_i \lambda_i s_i(a)$, where $\lambda_i = s(\imath_i a_i)$.

Therefore, if s an extremal state that is also σ-additive, from the previous decomposition we conclude that $s = s_i$ for a unique i.

Now assume that s is a state on E such that $s(\imath_i a_i) = 1$. Then $s(a_i) = 1/\imath_i$. Since $\imath_i a_i$ is a central element, for any $a \in E$, we have $s(a) = s(a \wedge \imath_i a_i) + s(a \wedge (\imath_i a_i)') = s(a \wedge \imath_i a_i) = s_i(a)$. $\square$

6 Conclusion

In the paper, we have studied effect algebras E which are also MV-effect algebras, i.e. every two elements of E are compatible. Since every MV-effect algebra satisfies the Riesz Decomposition Property, in other words, every two decompositions of the unit element 1 have a joint refinement, we have concentrated to effect algebras with RDP. We recall that RDP fails for $\mathcal{E}(H)$, and every effect algebra with RDP is an interval in an interpolation Abelian po-group with strong unit, and every MV-effect algebra is an interval in a lattice ordered group with strong unit.

The main result says, Theorem 4.8, that every σ-orthocomplete atomic effect algebra with RDP and with the countable set of atoms is in fact an MV-effect algebra which is the countable direct product of finite chains.

This results was applied also for pseudo-effect algebras, where it was proved, Theorem 5.2, that any analogous pseudo-effect algebra has to be commutative. In addition, the studied methods allow also to give a complete characterization of σ-additive states of our type of effect algebras, Theorem 5.3.

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